# Module-2: Analytic Functions-I

## 1 Introduction

The concept of analytic function is very important in complex analysis. In this module, we study some properties of analytic functions.

**Definition 1.** *A complex valued function $f(z)$ defined in a domain $D$ is said to be analytic in $D$ if $f(z)$ has a derivative at each point of $D$. The terms regular and holomorphic are also used with identical meanings in state of analytic.*

*A function $f(z)$ is said to be analytic at a point $z_0$ if it is analytic in some neighbourhood of $z_0$, i.e. if there exist a neighbourhood of $z_0$ at each point of which $f(z)$ is differentiable.*

*If $f(z)$ is not analytic at a point $z_0$, then $z_0$ is called a singular point of $f(z)$.*

**Remark 1.** *Real valued functions of complex variable are nowhere analytic unless these are constant valued. As for example, the functions $|z|$, $Re(z)$, $Im(z)$, etc. are nowhere analytic, since these are real valued but not constant on any domain in the complex plane.*

**Remark 2.** *The sum, difference and product of analytic functions are analytic in the same domain. The quotient $f(z)/g(z)$ of two analytic functions $f(z)$ and $g(z)$ is also analytic in a domain, provided $g(z) \neq 0$ in the domain. However, the sum etc. of non-analytic functions may be analytic.*

**Theorem 1.** *Let $f(z)$ be analytic in a domain $D$ such that $f(D) \subset D_1$. If a function $g$ is analytic in the domain $D_1$, and $F(z) = (gf)(z)$, then $F(z)$ is analytic in $D$.*

**Proof.** *Let $z_0 \in D$ is arbitrary. By Theorem* **??***, the function $F(z)$ is differentiable at $z_0$ and moreover,*

$$F'(z_0) = g'(f(z_0))f'(z_0).$$

*Since $z_0$ is an arbitrary point in $D$, it follows that $F(z)$ is differentiable in $D$ and hence analytic in $D$.*

**Example 1.** *Let $f(z) = \frac{1}{z-2}$ and $g(z) = z^3$. Discuss the analyticity of the composite function $gf$.*

**Solution.** *The function $f(z) = 1/(z-2)$ is analytic in the domain $D = \mathbb{C} \setminus \{2\}$ and $g(z)$ is analytic in $\mathbb{C}$. Now, $f(D) = \mathbb{C} \setminus \{0\}$ which is contained in $\mathbb{C}$, the domain of $g$. Therefore,*

$$\frac{d}{dz}[(gf)(z)] = g'(f(z))f'(z) = \frac{3}{(z-2)^2}\frac{-1}{(z-2)^2} = \frac{-3}{(z-2)^4}, \; z \neq 2.$$

**Example 2.** *If*

$$f(z) = \begin{cases} \frac{x^2y(y-ix)}{x^4+y^2}, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

*then prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any straight line but $f'(0)$ does not exist.*

**Solution.** *Suppose that $z \to 0$ along any straight line $y = mx$. Then*

$$\begin{aligned} \lim_{z\to 0} \frac{f(z)-f(0)}{z} &= \lim_{(x,y)\to(0,0)} \frac{x^2y(y-ix)}{(x+iy)(x^4+y^2)} \\ &= \lim_{(x,y)\to(0,0)} \frac{-ix^2y}{x^4+y^2} \\ &= \lim_{x\to 0} \frac{-imx^3}{x^2(x^2+m^2)} = 0. \end{aligned}$$

*Along the path $y = x^2$, we have*

$$\begin{aligned} \lim_{z\to 0} \frac{f(z)-f(0)}{z} &= \lim_{(x,y)\to(0,0)} \frac{-ix^2y}{x^4+y^2} \\ &= \lim_{x\to 0} \frac{-ix^4}{2x^4} = -\frac{i}{2}. \end{aligned}$$

*This shows that $f'(0)$ does not exist.*

### Entire function

A complex function $f(z)$ is said to be entire or integral if it is analytic in the entire complex plane. Any polynomial $P(z)$, $e^z$, $\sin z$, $\cos z$, etc. are examples of entire functions.

It is to be noted that the sum and the product of two or more entire functions are entire functions.

**Singular point**

If a function $f(z)$ fails to be analytic at a point $z_0$ but in every neighbourhood of $z_0$ there exist at least one point where the function is analytic, then the point $z_0$ is said to be a singular point or singularity of $f(z)$.

The function $f(z) = 1/z$ is analytic except for $z = 0$ and in each deleted neighbourhood of $z = 0$, $f(z)$ is analytic.Therefore by definition $z = 0$ is a singular point of $f(z) = 1/z$. For the function

$$f(z) = \frac{z^2+2}{z(z-1)^2(z+3)},$$

$z = 0$, $z = 1$ and $z = -3$ are singularities of $f(z)$, as the function is analytic except these points.

**Theorem 2.** *Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the first order partial derivatives of $u$ and $v$ at $(x_0, y_0)$ satisfy*

$$u_x = v_y \text{ and } u_y = -v_x. \tag{1}$$

**Proof.** *Given that $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Therefore*

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\ &= \lim_{(x,y)\to(x_0,y_0)} \frac{\{u(x,y)-u(x_0,y_0)\}+i\{v(x,y)-v(x_0,y_0)\}}{(x-x_0)+i(y-y_0)}. \end{aligned} \tag{2}$$

*Since $f'(z_0)$ exists, (2) must exists for all modes of approach of the point $(x, y)$ to $(x_0, y_0)$ and all the limiting values are same. Assuming $z \to z_0$ along a line parallel to real axis, i.e. $y = y_0$ and $x \to x_0$, we obtain from (2) that*

$$\begin{aligned} f'(z_0) &= \lim_{x\to x_0} \frac{\{u(x,y_0)-u(x_0,y_0)\}+i\{v(x,y_0)-v(x_0,y_0)\}}{x-x_0} \\ &= \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x\to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} \\ &= u_x(x_0,y_0)+iv_x(x_0,y_0). \end{aligned} \tag{3}$$

*Now letting $z \to z_0$ along a line parallel to imaginary axis, i.e. $x = x_0$ and $y \to y_0$, we obtain from (2) that*

$$\begin{aligned} f'(z_0) &= \lim_{y\to y_0} \frac{\{u(x_0,y)-u(x_0,y_0)\}+i\{v(x_0,y)-v(x_0,y_0)\}}{i(y-y_0)} \\ &= \lim_{y\to y_0} \frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \lim_{y\to y_0} \frac{v(x,y_0)-v(x_0,y_0)}{(y-y_0)} \\ &= -iu_y(x_0,y_0)+v_y(x_0,y_0). \end{aligned} \tag{4}$$

*Comparing (3) and (4) and equating the real and imaginary parts we obtain (1).*

**Note 1.** *The equation* $u_x = v_y$ *and* $u_y = -v_x$ *are known as Cauchy-Riemann equations (C-R equations).*

**Note 2.** *The C-R equations are the necessary conditions for the existence of derivative but these are not sufficient. In other words, if* $f(z)$ *is differentiable at* $z_0$*, then the C-R equations are satisfied at* $z_0$ *but the converse may not be true.*

**Note 3.** *In polar form the C-R equations can be written as*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

**Remark 3.** *We know that the pair of conjugate complex variable* $z$ *and* $\overline{z}$ *can be written as*

$$x = \frac{z+\overline{z}}{2}, \quad y = -i\left(\frac{z-\overline{z}}{2}\right).$$

*Now we introduce the following differential operators:*

$$\frac{\partial}{\partial z} := \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} := \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

*If* $f = u + iv$*, then it follows that*

$$\begin{aligned} f_z = \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}[(u_x + v_y) + i(v_x - u_y)]. \end{aligned}$$

*Similarly, we obtain*

$$\begin{aligned} f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]. \end{aligned}$$

*Therefore, the C-R equations are exactly equivalent to* $f_{\overline{z}}(z_0) = 0$ *which is also referred to as the complex form of the C-R equations.*

With this notation, we have the following alternate form of Theorem 2.

**Theorem 3.** *A necessary condition for a complex - valued function* $f = u + iv$ *to be differentiable at* $z_0$ *is that* $f_{\overline{z}}(z_0) = 0$.

The following example shows that the validity of C-R equations at a point is not sufficient to ensure the existence of the derivative at that point.

**Example 3.** *Let*

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

*Show that though C-R equations are satisfied at* $(0,0)$, $f'(0)$ *does not exist.*

**Solution.** *Let* $f(z) = u(x,y) + iv(x,y)$. *Then*

$$u(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

*and*

$$v(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

*Therefore*

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1.$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1.$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1.$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y}{y} = 1.$$

*Since* $u_x = v_y$ *and* $u_y = -v_x$, *C-R equations are satisfied at* $(0,0)$. *Now on* $y = mx$ *we have*

$$\begin{aligned} f'(0) &= \lim_{z\to 0} \frac{f(z) - f(0)}{z} \\ &= \lim_{(x,y)\to(0,0)} \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)(x+iy)} \\ &= \lim_{x\to 0} \frac{x^3(1-m^3) + ix^3(1+m^3)}{x^3(1+m^2)(1+im)}. \end{aligned}$$

*Since the value of the limit depends on* $m$, $f'(0)$ *does not exist.*

**Example 4.** *For the function defined by* $f(z) = \sqrt{|xy|}$, *show that the C-R equations are satisfied at* $z = 0$, *but the function is not differentiable at this point.*

**Solution.** *Considering* $f(z) = u(x,y)+iv(x,y)$ *we have* $u(x,y) = \sqrt{|\, xy\,|}$ *and* $v(x,y) = 0$. *We note that the given function* $f(z)$ *is identically zero on the real and imaginary axes. Therefore, it is trivial to see that*

$$u_x(0,0) \;=\; u_y(0,0) \;=\; v_x(0,0) \;=\; v_y(0,0) \;=\; 0.$$

*As for example,*

$$u_x(0,0) \;=\; \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} \;=\; \lim_{x\to 0} \frac{0}{x} \;=\; 0.$$

*Thus, C-R equations hold at* $z = 0$. *However, taking* $h = re^{i\theta} \neq 0$ *with* $r \to 0$, *we see that*

$$\begin{aligned} \lim_{h\to 0} \frac{f(h)-f(0)}{h} &= \lim_{r\to 0} \frac{|\, r^2 \sin\theta\cos\theta\,|^{1/2}}{r(\cos\theta + i\sin\theta)} \\ &= \frac{e^{-i\theta}\,|\sin 2\theta\,|^{1/2}}{\sqrt{2}} \end{aligned}$$

*which clearly depends upon the value of* $\theta$. *From this we can conclude that* $f$ *is not differentiable at* $z = 0$ , *though* $f$ *satisfies the C-R equations at the origin.*

**Example 5.** *Find the nature of C-R equations for the function* $f(z) = |\, z\,|^2$.

**Solution.** *Here* $f(z) = |\, z\,|^2 = x^2 + y^2$. *Therefore*

$$u(x,y) \;=\; x^2 + y^2 \;\; \textit{and} \;\; v(x,y) \;=\; 0.$$

*Differentiating we get*

$$u_x \;=\; 2x, \;\; u_y \;=\; 2y, \;\; v_x \;=\; 0, \;\; \textit{and} \;\; v_y \;=\; 0.$$

*Thus C-R equations are not satisfied unless* $x = y = 0$ *and hence* $f'(z)$ *does not exist at any point* $z \neq 0$.